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Dual-bounded generating problems: Efficient and inefficient points for discrete probability distributions and sparse boxes for multidimensional data[☆]

Leonid Khachiyan[a,✠], Endre Boros[b,*], Khaled Elbassioni[c], Vladimir Gurvich[b], Kazuhisa Makino[d]

[a] *Department of Computer Science, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ 08854-8003, United States*
[b] *RUTCOR, Rutgers University, 640 Bartholomew Road, Piscataway, NJ 08854-8003, United States*
[c] *Max-Planck-Institut für Informatik, Saarbrücken, Germany*
[d] *Department of Mathematical Informatics, Graduate School of Information and Technology, University of Tokyo, Tokyo, 113-8656, Japan*

Abstract

We show that $|\mathcal{X}| \le n|\mathcal{Y}|$ must hold for two finite sets $\mathcal{X}, \mathcal{Y} \subset \mathbb{R}^n$ whenever they can be separated by a nonnegative linear function such that $\mathcal{X}$ is above $\mathcal{Y}$ and the componentwise minimum of any two distinct points in $\mathcal{X}$ is dominated by some point in $\mathcal{Y}$. As a consequence, we obtain an incremental quasi-polynomial time algorithm for generating all maximal integer feasible solutions for a given monotone system of separable inequalities, for generating all p-inefficient points of a given discrete probability distribution, and for generating all maximal hyper-rectangles which contain a specified fraction of points of a given set in $\mathbb{R}^n$. This provides a substantial improvement over previously known exponential time algorithms for these generation problems related to Integer and Stochastic Programming, and Data Mining. Furthermore, we give an incremental polynomial time generation algorithm for monotone systems with fixed number of separable inequalities, implying that for discrete probability distributions with independent coordinates, both p-efficient and p-inefficient points can be separately generated in incremental polynomial time.

Keywords: Incremental generation; Maximal empty boxes; p-efficient points

1. Introduction

Let $\mathcal{X}$ and $\mathcal{Y}$ be two finite sets of points in $\mathbb{R}^n$ such that

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* Corresponding address: Rutgers University, The State University of New Jersey, RUTCOR, 640 Bartolomew Road, 08854-8003 Piscataway, NJ, United States. Tel.: +1 7324453235; fax: +1 7324455472.

E-mail addresses: boros@rutcor.rutgers.edu (E. Boros), elbassio@mpi-sb.mpg.de (K. Elbassioni), gurvich@rutcor.rutgers.edu (V. Gurvich), makino@mist.i.u-tokyo.ac.jp (K. Makino).

✠ Our friend and co-author, Leonid Khachiyan passed away with tragic suddenness, while we were working on the final version of this paper.

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(P1) $\mathcal{X}$ and $\mathcal{Y}$ can be separated by a nonnegative linear function: $w(x) > t \geq w(y)$ for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, where $t \in \mathbb{R}$ is a real threshold, and $w(x) = \sum_{i=1}^{n} w_i x_i$, for some nonnegative weights $w_1, \ldots, w_n \in \mathbb{R}_+$.

(P2) For any two distinct points $x, x' \in \mathcal{X}$, their componentwise minimum $x \wedge x'$ is dominated by some $y \in \mathcal{Y}$, i.e. $x \wedge x' \leq y$.

Given $\mathcal{X}, \mathcal{Y} \subseteq \mathbb{R}^n$ satisfying properties (P1) and (P2), one may ask how large the size of $\mathcal{X}$ can be in terms of the size of $\mathcal{Y}$. For instance, if $\mathcal{X}$ is the set of the n-dimensional unit vectors, and $\mathcal{Y} = \{\mathbf{0}\}$ is the set containing only the origin, then $\mathcal{X}$ and $\mathcal{Y}$ satisfy properties (P1), (P2), and the ratio between their cardinalities is n. We shall show that this actually is an extremal case:

Lemma 1 (*Intersection Lemma*). *If $\mathcal{X}$ and $\mathcal{Y} \neq \emptyset$ are two finite sets of points in $\mathbb{R}^n$ satisfying properties* (P1) *and* (P2) *above, then*

$$|\mathcal{X}| \leq n|\mathcal{Y}|. \tag{1}$$

An analogous statement for binary sets $\mathcal{X}, \mathcal{Y} \subseteq \{0, 1\}^n$ was shown in [6]. Let us also recall from [6] that condition (P1) is essential, since without that $|\mathcal{X}|$ could be exponentially larger than $|\mathcal{Y}|$ already in the binary case. Let us also remark that the nonnegativity of the weight vector w is also necessary. Consider for instance $\mathcal{Y} = \{(1, 1, \ldots, 1)\}$ and an arbitrary number of points in the set $\mathcal{X}$ such that $0 \leq x_i < 1$ for all $x \in \mathcal{X}$ and $i = 1, \ldots, n$. Then clearly (P2) holds, and (P1) is satisfied with $w = (-1, 0, \ldots, 0)$ and $t = -1$. However, it is impossible to bound in this case the cardinality of $\mathcal{X}$ in terms of n and $|\mathcal{Y}| = 1$.

Let us further note that, due to the strict separation in (P1), we may assume without loss of generality that all weights are positive $w > 0$. In fact, it is enough to prove the lemma for $w = (1, 1, \ldots, 1)$, since scaling the ith coordinates of all points in $\mathcal{X} \cup \mathcal{Y}$ by $w_i > 0$ for $i = 1, \ldots, n$ always transforms the input into one satisfying (P1) with $w = (1, 1, \ldots, 1)$. Clearly, such scaling preserves the relative order with respect to each coordinate of the points, and scales properly their componentwise minimum, so that the transformed point sets will satisfy (P2) as well.

We prove Lemma 1 in Section 5. As a consequence of the lemma, we obtain new results on the complexity of several generation problems, including:

Monotone systems of separable inequalities: Given a system of inequalities on sums of single-variable monotone functions, generate all maximal feasible integer solutions of the system.

p-Efficient and p-inefficient points of discrete probability distributions: Given a random variable $\xi \in \mathbb{Z}^n$, generate all p-inefficient points, i.e. maximal vectors $x \in \mathbb{Z}^n$ whose cumulative probability $\Pr[\xi \leq x]$ does not exceed a certain threshold p, and/or generate all p-efficient points, i.e. minimal vectors $x \in \mathbb{Z}^n$ for which $\Pr[\xi \leq x] \geq p$. This problem has applications in Stochastic Programming [8,19].

Maximal k-boxes: Given a set of points in $\mathbb{R}^n$ and a nonnegative integer k, generate all maximal n-dimensional intervals (*boxes*), each of which contains at most k of the given points in its interior. Such intervals are called empty boxes or empty rectangles, when $k = 0$. This problem has applications in computational geometry, data mining and machine learning [1,2,7,9,14,15,17,18].

These problems are described in more detail in the following sections. What they have in common is that each can be modeled by a property π over a set of vectors $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2 \times \cdots \times \mathcal{C}_n$, where $\mathcal{C}_i$, $i = 1, \ldots, n$ are finite subsets of the reals, and π is anti-monotone, i.e., if $x, y \in \mathcal{C}$, $x \geq y$, and x satisfies property π, then y also satisfies π. Each problem in turn can be stated as that of incrementally generating the family $\mathcal{F}_\pi$ of all *maximal* elements of $\mathcal{C}$ satisfying π:

GEN$(\mathcal{F}_\pi, \mathcal{E})$: *Given an anti-monotone property π, and a subfamily $\mathcal{E} \subseteq \mathcal{F}_\pi$ of the maximal elements satisfying π, either find a new maximal element $x \in \mathcal{F}_\pi \setminus \mathcal{E}$, or prove that $\mathcal{E} = \mathcal{F}_\pi$.*

Clearly, the entire family $\mathcal{F}_\pi$ can be generated by initializing $\mathcal{E} = \emptyset$ and iteratively solving the above problem $|\mathcal{F}_\pi|+1$ times.

For a subset $\mathcal{A} \subseteq \mathcal{C}$, denote by $\mathcal{I}(\mathcal{A})$ the set of maximal independent elements of $\mathcal{A}$, i.e. the set of those elements $x \in \mathcal{C}$ that are maximal with respect to the property that $x \not\geq a$ for all $a \in \mathcal{A}$. Let $\mathcal{I}^{-1}(\mathcal{A})$ be the set of elements $x \in \mathcal{C}$ that are minimal with the property that $x \not\leq a$ for all $a \in \mathcal{A}$. In particular, $\mathcal{I}^{-1}(\mathcal{F}_\pi)$ denotes the family of *minimal* elements of $\mathcal{C}$ which *do not* satisfy property π.

Following [6], let us call $\mathcal{F}_\pi$ *uniformly dual-bounded*, if for every non-empty subfamily $\mathcal{E} \subseteq \mathcal{F}_\pi$ we have

$$|\mathcal{I}^{-1}(\mathcal{E}) \cap \mathcal{I}^{-1}(\mathcal{F}_\pi)| \leq p(|\pi|, n, |\mathcal{E}|) \tag{2}$$

for some polynomial $p(\cdot)$, where $|\pi|$ denotes the length of the description of property π. It is known that for uniformly dual-bounded families $\mathcal{F}_\pi$ of subsets of a discrete box $\mathcal{C}$ problem GEN($\mathcal{F}_\pi, \mathcal{E}$) can be reduced in polynomial time to the following *dualization* problem on boxes (see [5] and also [4,12,13]):

DUAL($\mathcal{C}, \mathcal{A}, \mathcal{B}$): *Given an integer box $\mathcal{C}$, a family of vectors $\mathcal{A} \subseteq \mathcal{C}$ and a subset $\mathcal{B} \subseteq \mathcal{I}(\mathcal{A})$ of its maximal independent vectors, either find a new maximal independent vector $x \in \mathcal{I}(\mathcal{A}) \setminus \mathcal{B}$, or prove that no such vector exists, that is that $\mathcal{B} = \mathcal{I}(\mathcal{A})$.*

It is furthermore known that problem DUAL($\mathcal{C}, \mathcal{A}, \mathcal{B}$) can be solved in $poly(n) + m^{o(\log m)}$ time, where $m = |\mathcal{A}| + |\mathcal{B}|$ (see [5,11]). However, it is still open whether DUAL($\mathcal{C}, \mathcal{A}, \mathcal{B}$) has a polynomial time algorithm (see e.g., [4,10,11,16]).

For each of the problems described above, it will be shown that the families $\mathcal{I}^{-1}(\mathcal{E}) \cap \mathcal{I}^{-1}(\mathcal{F}_\pi)$ and $\mathcal{E} \subseteq \mathcal{F}_\pi$ can be related to two sets of points $\mathcal{X}, \mathcal{Y}$ satisfying the conditions of Lemma 1. Then the lemma will imply (2), which in its turn is sufficient for the efficient generation of the family $\mathcal{F}_\pi$ (see [5]).

In particular, it will follow that each of the above generation problems can be solved incrementally in *quasi-polynomial time*. Furthermore, we give incremental *polynomial-time* algorithms for generating

- all maximal feasible, and separately, all minimal infeasible integer vectors for systems with fixed number of monotone separable inequalities, and
- all p-efficient, and separately, all p-inefficient points of discrete probability distributions with independent coordinates

In the last section, we consider some generalizations of the intersection lemma. Namely, we show that an analogous lemma holds for families of vectors in the product of arbitrary meet semi-lattices. As an application, we obtain quasi-polynomial time algorithms for generating maximal feasible solutions for systems of monotone inequalities on sums of separable functions with bounded number of variables, and for generating maximal k-boxes whose diameter does not exceed a given threshold, for a given set of points.

2. Systems of monotone separable inequalities

For $i = 1, 2, \ldots, n$, let l_i and u_i be given integers with $l_i \leq u_i$, and let $\mathcal{C}_i \stackrel{\text{def}}{=} \{l_i, l_i + 1, \ldots, u_i\}$. A function $f : \mathcal{C}_i \longrightarrow \mathbb{R}$ is called *monotone* if, for $x, y \in \mathcal{C}_i$, $f(x) \geq f(y)$ whenever $x \geq y$. Let $f_{ij} : \mathcal{C}_i \longrightarrow \mathbb{R}$, $i = 1, 2, \ldots, n$, $j = 1, \ldots, r$ be polynomial-time computable monotone functions, and consider the system of inequalities

$$\sum_{i=1}^{n} f_{ij}(x_i) \leq t_j, \quad j = 1, \ldots, r, \tag{3}$$

over the elements $x \in \mathcal{C} = \{x \in \mathbb{Z}^n \mid l \leq x \leq u\}$, where $l = (l_1, \ldots, l_n)$, $u = (u_1, \ldots, u_n)$, and $t = (t_1, \ldots, t_r)$ is a given r-dimensional real vector.

Let us denote by $\mathcal{F}_t$ the set of all maximal feasible solutions for (3). Then $\mathcal{I}^{-1}(\mathcal{F}_t)$ represents the set of all minimal infeasible vectors for (3).

Generalizing results on monotone systems of *linear* inequalities from [5], we will now use Lemma 1 to prove the following:

Theorem 1. *If $\mathcal{F}_t$ is the family of all maximal feasible solutions of (3), and $\mathcal{E} \subseteq \mathcal{F}_t$ is non-empty, then*

$$\left|\mathcal{I}^{-1}(\mathcal{E}) \cap \mathcal{I}^{-1}(\mathcal{F}_t)\right| \leq rn|\mathcal{E}|. \tag{4}$$

In particular, $|\mathcal{I}^{-1}(\mathcal{F}_t)| \leq rn|\mathcal{F}_t|$.

Proof. For a given index $j \in \{1, \ldots, r\}$, let us define a monotone mapping $\phi_j : \mathcal{C} \longrightarrow \mathbb{R}^n$ by setting $\phi_j(x) = (f_{1j}(x_1), \ldots, f_{nj}(x_n))$ for $x \in \mathcal{C}$. Let $\mathcal{Y}_j = \{\phi_j(x) \mid x \in \mathcal{E}\}$, and let $\mathcal{X}_j = \{\phi_j(x) \mid x \in \mathcal{I}^{-1}(\mathcal{E}), \sum_{i=1}^n f_{ij}(x_i) > t_j\}$. In other words, $\mathcal{X}_j$ is the ϕ_j-image of those minimal infeasible solutions of (3) in $\mathcal{I}^{-1}(\mathcal{E})$ which violate the jth inequality. Since the functions f_{ij} are monotone, and since we consider only maximal feasible or minimal infeasible vectors for (3), the mappings $\mathcal{E} \longrightarrow \mathcal{Y}_j$ and $\{x \in \mathcal{I}^{-1}(\mathcal{E}) \mid \sum_{i=1}^n f_{ij}(x_i) > t_j\} \longrightarrow \mathcal{X}_j$ are one-to-one.

It is easy to see that the sets $\mathcal{X}_j$ and $\mathcal{Y}_j$ satisfy the conditions of Lemma 1 with $w = (1, 1, \ldots, 1)$ and $t = t_j$, and hence $|\mathcal{X}_j| \le n|\mathcal{Y}_j| = n|\mathcal{E}|$ by Lemma 1. Now (4) follows from the fact that $\mathcal{I}^{-1}(\mathcal{E}) \cap \mathcal{I}^{-1}(\mathcal{F}_t) = \bigcup_{j=1}^r \{x \in \mathcal{I}^{-1}(\mathcal{E}) \mid \sum_{i=1}^n f_{ij}(x_i) > t_j\}$. □

Since by (4) the family $\mathcal{F}_t$ is uniformly dual-bounded, the results of [5], as we cited earlier, directly imply the following.

Corollary 1. *Given a partial list $\mathcal{E} \subseteq \mathcal{F}_t$ of maximal feasible solutions for (3), problem GEN($\mathcal{F}_t, \mathcal{E}$) can be solved in $k^{o(\log k)}$ time, where $k = \max\{n, r, |\mathcal{E}|\}$, using $poly(k) \log(\|u - l\|_\infty + 1)$ feasibility tests for (3).*

It should be mentioned that in contrast to (4), the size of $\mathcal{F}_t$ cannot be bounded by a polynomial in n, r, and $|\mathcal{I}^{-1}(\mathcal{F}_t)|$, even for monotone systems of linear inequalities (see e.g. [5]). However, for systems (3) with *constant* r, we shall show that such a bound exists, and further that the generation problem can be solved in polynomial time:

Theorem 2. *If $\mathcal{F}_t$ is the family of maximal feasible solutions of (3), and $\mathcal{E} \subseteq \mathcal{I}^{-1}(\mathcal{F}_t)$ is non-empty, then*

$$|\mathcal{I}(\mathcal{E}) \cap \mathcal{F}_t| \le (n|\mathcal{E}|)^r. \tag{5}$$

In particular, $|\mathcal{F}_t| \le \left(n|\mathcal{I}^{-1}(\mathcal{F}_t)|\right)^r$.

Theorem 3. *If the number of inequalities in (3) is bounded, then both the maximal feasible and minimal infeasible vectors can be generated in incremental time, polynomial in n, r and* $\log(\|u - l\|_\infty + 1)$.

The proofs of Theorems 2 and 3 will be given in Section 6. In the next section, we consider an application of Theorem 3 for the case of $r = 1$.

3. *p*-Efficient and *p*-inefficient points of probability distributions

Let ξ be an n-dimensional random variable on $\mathbb{Z}^n$, with a finite support $\mathcal{S} \subseteq \mathbb{Z}^n$, i.e., $\sum_{q \in \mathcal{S}} \Pr[\xi = q] = 1$, and $\Pr[\xi = q] > 0$ for $q \in \mathcal{S}$. Given a threshold probability $p \in (0, 1)$, a point $x \in \mathbb{Z}^n$ is said to be *p-efficient* if it is minimal with the property that $\Pr[\xi \le x] > p$. Let us conversely say that $x \in \mathbb{Z}^n$ is *p-inefficient* if it is maximal with the property that $\Pr[\xi \le x] \le p$. Denote respectively by $\mathcal{F}_{\mathcal{S},p}$ and $\mathcal{I}^{-1}(\mathcal{F}_{\mathcal{S},p})$ the sets of all p-inefficient and p-efficient points for ξ. Clearly, these sets are finite since, in each dimension $i \in [n] \overset{\text{def}}{=} \{1, \ldots, n\}$, we need to consider only the projections $\mathcal{C}_i \overset{\text{def}}{=} \{q_i, q_i - 1 \mid q \in \mathcal{S}\} \subseteq \mathbb{Z}$. In other words, the sets $\mathcal{F}_{\mathcal{S},p}$ and $\mathcal{I}^{-1}(\mathcal{F}_{\mathcal{S},p})$ can be regarded as subsets of a finite integral box $\mathcal{C} = \mathcal{C}_1 \times \cdots \times \mathcal{C}_n$ of size at most $2|\mathcal{S}|$ along each dimension.

Theorem 4. *Given a partial list $\mathcal{E} \subseteq \mathcal{F}_{\mathcal{S},p}$ of p-inefficient points, problem GEN($\mathcal{F}_{\mathcal{S},p}, \mathcal{E}$) can be solved in $k^{o(\log k)}$ time, where $k \overset{\text{def}}{=} \max\{n, |\mathcal{S}|, |\mathcal{E}|\}$.*

Proof. This statement is again a consequence of the fact that the set $\mathcal{F}_{\mathcal{S},p}$ is uniformly dual-bounded. Specifically, we can show that

$$\left|\mathcal{I}^{-1}(\mathcal{E}) \cap \mathcal{I}^{-1}(\mathcal{F}_{\mathcal{S},p})\right| \le |\mathcal{S}||\mathcal{E}| \tag{6}$$

holds for any non-empty subset $\mathcal{E} \subseteq \mathcal{F}_{\mathcal{S},p}$. To see (6), let $\mathcal{X} = \{\phi(x) \mid x \in \mathcal{I}^{-1}(\mathcal{E}) \cap \mathcal{I}^{-1}(\mathcal{F}_{\mathcal{S},p})\}$ and $\mathcal{Y} = \{\phi(y) \mid y \in \mathcal{E}\}$, where $\phi : \mathbb{Z}^n \longrightarrow \mathbb{R}^{|\mathcal{S}|}$ is the mapping defined by $\phi_q(x) = \Pr[\xi = q]$ for $q \in \mathcal{S}$ with $q \le x$, and $\phi_q(x) = 0$ for $q \in \mathcal{S}$ with $q \not\le x$. One can easily check that the mapping ϕ is one-to-one between $\mathcal{X}$ and $\mathcal{I}^{-1}(\mathcal{E}) \cap \mathcal{I}^{-1}(\mathcal{F}_{\mathcal{S},p})$, and that the families $\mathcal{X}$ and $\mathcal{Y}$ satisfy properties (P1) and (P2) with $w = (1, 1, \ldots, 1)$ and $t = p$. Therefore, (6) follows from the intersection lemma. □

In particular, all p-inefficient points of a discrete probability distribution can be enumerated incrementally in quasi-polynomial time. In general, a result analogous to that for p-efficient points is highly unlikely to hold, since the problem is NP-hard:

Proposition 1. *Given a discrete random variable ξ on a finite support set $\mathcal{S} \subseteq \mathbb{R}^n$, a threshold probability $p \in (0, 1)$, and a partial list $\mathcal{E} \subseteq \mathcal{I}^{-1}(\mathcal{F}_{\mathcal{S},p})$ of p-efficient points for ξ, it is* NP*-complete to decide if $\mathcal{E} \neq \mathcal{I}^{-1}(\mathcal{F}_{\mathcal{S},p})$.*

Proof. Consider the well-known NP-complete problem of deciding whether a given graph $G = (V, E)$ contains an independent set of size at least t, where $t \geq 2$ is a given threshold. Let $\mathcal{S} \subseteq \{0, 1\}^V$ be the set of points consisting of the $|V|$ incidence vectors of the vertices of G, and $t - 2$ copies of each of the $|E|$ incidence vectors of the edges. Let ξ be an n-dimensional integer-valued random variable having uniform distribution on $\mathcal{S}$, i.e. $\Pr[\xi = q] = 1/|\mathcal{S}|$ if and only if $q \in \mathcal{S}$. Then, for $p = (t - 1)/|\mathcal{S}|$, the incidence vector of each edge is a p-efficient point for ξ, and it is easy to see that there is another p-efficient point if and only if there is an independent set of G of size at least t. □

Finally we observe that if ξ is an integer-valued finite random variable with independent coordinates $\xi_1, \ldots, \xi_n$, then the generation of both $\mathcal{I}^{-1}(\mathcal{F}_{\mathcal{S},p})$ and $\mathcal{F}_{\mathcal{S},p}$ can be done in polynomial time, even if the number of points $\mathcal{S}$, defining the distribution of ξ, is *exponential* in n (but provided that the distribution function for each component ξ_i is computable in polynomial-time). Indeed, by independence we have $\Pr[\xi \leq x] = \prod_{i=1}^{n} \Pr[\xi_i \leq x_i]$. Defining $f(x) = \log \Pr[\xi \leq x] = \sum_{i=1}^{n} \log \Pr[\xi_i \leq x_i]$, we can write $f(x)$ as the sum of single-variable monotone functions $f_1, \ldots, f_n$, where $f_i = \log \Pr[\xi_i \leq x_i]$, for $i = 1, \ldots, n$, and where we regard $\log 0$ as $-\infty$. Let $l_i = \min\{x_i \in \mathbb{Z} \mid \Pr[\xi_i \leq x_i] > 0\} - 1$, $u_i = \min\{x_i \in \mathbb{Z} \mid \Pr[\xi_i \leq x_i] = 1\}$, and $\mathcal{C}_i = \{z \in \mathbb{Z} \mid l_i \leq z \leq u_i\}$. Then the p-inefficient (p-efficient) points are the maximal feasible (respectively, minimal infeasible) solutions of the monotone separable inequality $\sum_{i=1}^{n} f_i(x_i) \leq t \stackrel{\text{def}}{=} \log p$ over the product space $\mathcal{C} \stackrel{\text{def}}{=} \mathcal{C}_1 \times \cdots \times \mathcal{C}_n$. Consequently, Theorem 3 immediately yields the following:

Corollary 2. *If the coordinates of a random variable ξ over $\mathbb{Z}^n$ are independent, then both the p-efficient and the p-inefficient points for ξ can be enumerated in incremental polynomial time.*

4. Maximal *k*-boxes

Let $\mathcal{S}$ be a set of points in $\mathbb{R}^n$, and $k \leq |\mathcal{S}|$ be a given integer. A *maximal k-box* is a closed n-dimensional interval which contains at most k points of $\mathcal{S}$ in its interior, and which is maximal with respect to this property (i.e. cannot be extended in any direction without strictly enclosing more points of $\mathcal{S}$). Let $\mathcal{F}_{\mathcal{S},k}$ be the set of all maximal k-boxes. Let us note that without any loss of generality, we could consider the generation of the boxes $\{B \cap D \mid B \in \mathcal{F}_{\mathcal{S},k}\}$, where D is a fixed bounded box containing all points of $\mathcal{S}$ in its interior. Let us further note that the ith coordinate of each vertex of such a box is the same as p_i for some $p \in \mathcal{S}$, or the ith coordinate of a vertex of D, hence all these coordinates belong to a finite set of cardinality at most $|\mathcal{S}| + 2$. In what follows we shall view $\mathcal{F}_{\mathcal{S},k}$ as a set of boxes with vertices belonging to such a finite grid.

The problem of generating all elements of $\mathcal{F}_{\mathcal{S},0}$ has been studied in the machine learning and computational geometry literatures (see [9,14,15], and also [2,7,17,18]), and is motivated by the discovery of missing associations or “holes” in data mining applications (see [1,14,15]). All known algorithms that solve this problem have running time complexity exponential in the dimension n of the given point set. In contrast, we show in this paper that the problem can be solved in quasi-polynomial time:

Theorem 5. *Given a point set $\mathcal{S} \subseteq \mathbb{R}^n$, an integer k, and a partial list of maximal empty boxes $\mathcal{E} \subseteq \mathcal{F}_{\mathcal{S},k}$, problem GEN($\mathcal{F}_{\mathcal{S},k}, \mathcal{E}$) can be solved in $m^{o(\log m)}$ time, where $m \stackrel{\text{def}}{=} \max\{n, |\mathcal{S}|, |\mathcal{E}|\}$.*

Proof. Let us define $\mathcal{C}_i = \{p_i \mid p \in \mathcal{S}\}$ for $i = 1, \ldots, n$ and consider the family of boxes $\mathcal{B} = \{[a, b] \subseteq \mathbb{R}^n \mid a, b \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_n,\ a \leq b\}$. For $i = 1, \ldots, n$, let $u_i = \max \mathcal{C}_i$, and let $\mathcal{C}_i^* \stackrel{\text{def}}{=} \{u_i - p \mid p \in \mathcal{C}_i\}$ be the chain ordered in the direction opposite to $\mathcal{C}_i$. Consider the $2n$-dimensional box $\mathcal{C} = \mathcal{C}_1^* \times \cdots \times \mathcal{C}_n^* \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_n$ and let us represent every n-dimensional interval $[a, b] \in \mathcal{B}$ as the $2n$-dimensional vector $(u - a, b) \in \mathcal{C}$, where $u = (u_1, \ldots, u_n)$. This gives a monotone injective mapping $\mathcal{B} \longrightarrow \mathcal{C}$ (not all elements of $\mathcal{C}$ define a box, since $a_i > b_i$ is possible for $(u - a, b) \in \mathcal{C}$).

Let us now define the anti-monotone property π to be satisfied by an $x \in \mathcal{C}$ if and only if x does not define a box, or the box defined by x contains at most k points of S in its interior. Then the set $\mathcal{F}_{\mathcal{S},k}$ can be identified with

$\mathcal{F}_\pi \subseteq \mathcal{B} \subseteq \mathcal{C}$, and for any non-empty family $\mathcal{E} \subseteq \mathcal{F}_{\mathcal{S},k}$, the set $\mathcal{I}^{-1}(\mathcal{E}) \cap \mathcal{I}^{-1}(\mathcal{F}_{\mathcal{S},k})$ consists of all those minimal boxes of $x \in \mathcal{B} \subseteq \mathcal{C}$ which contain at least $k+1$ points of $\mathcal{S}$ in their interior and have the property that any of their immediate predecessors $x' \le x$ in $\mathcal{C}$ is dominated by some $y \in \mathcal{E}$.

Finally, consider the sets $\mathcal{X} = \{\phi(x) \mid x \in \mathcal{I}^{-1}(\mathcal{E}) \cap \mathcal{I}^{-1}(\mathcal{F}_{\mathcal{S},k})\}$ and $\mathcal{Y} = \{\phi(y) \mid y \in \mathcal{E}\}$, where $\phi(x) \in \{0,1\}^{\mathcal{S}}$ is the characteristic vector of the subset of $\mathcal{S}$ contained in the interior of the box defined by $x \in \mathcal{C}$. Since there is exactly one minimal box containing a given non-empty set $S' \subseteq S$ in its interior, the mapping ϕ is one-to-one between $\mathcal{X}$ and $\mathcal{I}^{-1}(\mathcal{E}) \cap \mathcal{I}^{-1}(\mathcal{F}_{\mathcal{S},k})$. It is also easy to see that the sets $\mathcal{X}$ and $\mathcal{Y}$ satisfy properties (P1) and (P2) with $w = (1, 1, \ldots, 1)$ and $t = k$. Hence

$$|\mathcal{I}^{-1}(\mathcal{E}) \cap \mathcal{I}^{-1}(\mathcal{F}_{\mathcal{S},k})| \le |\mathcal{S}||\mathcal{E}| \tag{7}$$

follows by applying the intersection lemma. (Note that for $k = 0$, we have the stronger inequality $|\mathcal{I}^{-1}(\mathcal{F}_{\mathcal{S},0})| = |\mathcal{S}|$).

Since the family $\mathcal{F}_{\mathcal{S},k} = \mathcal{F}_\pi$ is uniformly dual-bounded, Theorem 5 follows from the complexity bound for the dualization problem on boxes stated in the introduction. □

Theorem 5 should be contrasted with the following negative result:

Proposition 2. *Given a set of points $\mathcal{S} \subseteq \mathbb{Z}^n$ and an integer $k \le |\mathcal{S}|$, let us consider the family $\mathcal{B}_{\mathcal{S},k}$ of all minimal boxes having integral vertices, each of which contains at least k points of $\mathcal{S}$ in its (strict) interior. Let further $\mathcal{X}$ be a subfamily $\mathcal{X} \subseteq \mathcal{B}_{\mathcal{S},k}$ of such minimal boxes. Then it is NP-complete to decide if $\mathcal{X} \neq \mathcal{B}_{\mathcal{S},k}$.*

Proof. We show that the problem is polynomial-time reducible to checking whether a given graph $G = (V, E)$ contains an independent set of size at least t, where $2 \le t \le |V|$ is a given threshold. Let $\mathcal{S} \subseteq \{0, 2\}^V$ be the set of points consisting of the double of the $|V|$ incidence vectors of the vertices of G, $t-2$ copies of the double of each of the $|E|$ incidence vectors of the edges, and $|V| + (t-2)|E| + 1$ copies of the origin $(0, \ldots, 0)$. Let $k = t + |V| + (t-2)|E| + 1$. Then to each edge $e = \{i, j\} \in E$ we can associate a minimal box $[a, b]$, containing k points of $\mathcal{S}$ in its interior, with lower point $a = (-1, \ldots, -1)$, and upper point having $b_i = b_j = 3$, and $b_r = 1$ for $r \neq i, j$. It is furthermore easy to see that there is another minimal box containing at least k points of $\mathcal{S}$ in its interior if and only if there is an independent set of G of size at least t. □

5. Proof of the intersection lemma

As mentioned in the introduction, we may assume without loss of generality that all the weights are 1's. We can further assume that $|\mathcal{X}| \ge 1$ and that $\mathcal{Y}$ is an *inclusion-wise* minimal family, each vector of which is *component-wise* minimal for properties (P1) and (P2). For $i = 1, \ldots, n$, let $l_i \stackrel{\text{def}}{=} \min\{x_i \mid x \in \mathcal{X}\}$, and $u_i \stackrel{\text{def}}{=} \max\{x_i \mid x \in \mathcal{X}\}$.

To prove the lemma, we shall show by induction on $|\mathcal{X}|$ that

$$|\mathcal{X}| \le \sum_{y \in \mathcal{Y}} q(y), \tag{8}$$

where $q(y)$ is the number of components y_i such that $y_i < u_i$.

For $|\mathcal{X}| = 1$ the statement is true since $\mathcal{Y}$ is non-empty and $q(y) = 0$ for $y \in \mathcal{Y}$ implies by (P1) that $\mathcal{X} = \emptyset$. Let us assume therefore that $|\mathcal{X}| \ge 2$, and define for every $i = 1, \ldots, n$ and $z \in \mathbb{R}$ the families

$$\mathcal{X}(i, z) = \{x \in \mathcal{X} \mid x_i \ge z\}, \qquad \mathcal{Y}(i, z) = \{y \in \mathcal{Y} \mid y_i \ge z\}.$$

Clearly, these families satisfy conditions (P1) and (P2). Furthermore, we may assume without loss of generality that $\mathcal{Y}(i, z) = \emptyset$ implies $\mathcal{X}(i, z) = \emptyset$ for all $i \in [n]$ and $z \in \mathbb{R}$. Indeed, by (P2), if $|\mathcal{Y}(i, z)| = 0$ then $|\mathcal{X}(i, z)| \in \{0, 1\}$. If there is an $i \in [n]$ and $z \in \mathbb{R}$, such that $\mathcal{X}(i, z) = \{x\}$ and $\mathcal{Y}(i, z) = \emptyset$, then deleting the element x from $\mathcal{X}$ reduces $|\mathcal{X}|$ by 1 and reduces the sum $\sum_{y \in \mathcal{Y}} q(y)$ by at least 1.

Thus, we can assume by induction on the number of elements in $\mathcal{X}$ that

$$|\mathcal{X}(i, z)| \le \sum_{y \in \mathcal{Y}(i,z)} q(y) \tag{9}$$

whenever $|\mathcal{X}(i,z)| < |\mathcal{X}|$. Since the latter condition is satisfied for $z > l_i$, we can sum up inequalities (9), for all values $z > l_i$, and for all indices $i \in [n]$, to obtain

$$\sum_{i=1}^{n} \int_{z>l_i} |\mathcal{X}(i,z)| \mathrm{d}z \le \sum_{i=1}^{n} \int_{z>l_i} \sum_{y \in \mathcal{Y}(i,z)} q(y) \mathrm{d}z. \tag{10}$$

It is easily seen that the left hand side of (10) is equal to

$$L = \sum_{x \in \mathcal{X}} \sum_{i=1}^{n} (x_i - l_i),$$

while the right hand side is equal to

$$R = \sum_{y \in \mathcal{Y}} q(y) \sum_{i=1}^{n} (y_i - l_i).$$

Thus, we get by (P1) and (10) that

$$\left(t - \sum_{i=1}^{n} l_i\right) |\mathcal{X}| < L \le R \le \left(t - \sum_{i=1}^{n} l_i\right) \sum_{y \in \mathcal{Y}} q(y). \tag{11}$$

Note that $t - \sum_{i=1}^{n} l_i > 0$ can be assumed without loss of generality. Indeed, if $t \le \sum_{i=1}^{n} l_i$ then for an arbitrary $y \in \mathcal{Y}$ ($\mathcal{Y} \ne \emptyset$) we have $\sum_{i=1}^{n} y_i \le t \le \sum_{i=1}^{n} l_i$ by (P1). By the minimality of $\mathcal{Y}$, we must have $y_i \ge l_i$, for all $i = 1, \ldots, n$, implying that $t = \sum_{i=1}^{n} l_i$. But then $\mathcal{Y} = \{l\}$ and we can replace t by $t + \epsilon$, for a sufficiently small $\epsilon > 0$, and still satisfy property (P1). Thus inequality (8) follows from (11). □

6. Proof of Theorems 2 and 3

For $j = 1, 2, \ldots, r$, let $f_j(x) = \sum_{i=1}^{n} f_{ij}(x_i)$, where $x \in \mathcal{C} = \{x \in \mathbb{Z}^n \mid l_i \le x_i \le u_i, i = 1, 2, \ldots, n\}$. For a given real vector $t = (t_1, \ldots, t_r)$, let $\mathcal{F}_t$ be the set of all maximal feasible solutions of system (3).

For each $i \in [n] \stackrel{\text{def}}{=} \{1, \ldots, n\}$, let $\Delta_{ij} : \{l_i - 1, l_i, \ldots, u_i\} \to \mathbb{R}$ be the *difference* of f_{ij} defined by

$$\Delta_{ij}(x_i) = \begin{cases} f_{ij}(x_i + 1) - f_{ij}(x_i) & \text{if } x_i \in \{l_i, l_i + 1, \ldots, u_i - 1\} \\ +\infty & \text{if } x_i \in \{l_i - 1, u_i\}. \end{cases} \tag{12}$$

Let us now define, for each $j \in [r]$, a mapping μ^j from pairs of a vector $x \in \mathcal{C}$ and a component $i \in [n]$ with $x_i > l_i$ to vectors $y \in \mathcal{C}$ by

$$\mu^j(x,i)_k = \begin{cases} x_k - 1 & \text{if } k = i \\ x_k + \alpha_k & \text{otherwise,} \end{cases} \tag{13}$$

where $\alpha_k = \alpha_k(x,i,j)$ is a non-negative integer such that $\Delta_{kj}(x_k + \alpha_k) \ge \Delta_{ij}(x_i - 1)$ and $\Delta_{kj}(x_k + s) < \Delta_{ij}(x_i - 1)$ for all $s = 0, 1, \ldots, \alpha_k - 1$. Note that such α_k always exists by our definition (12).

Given any $x \in \mathcal{I}^{-1}(\mathcal{F}_t)$, there exists an index $j = \rho(x) \in [r]$ such that x violates the jth inequality of the system, i.e. $f_j(x) > t_j$. For $\mathcal{E} \subseteq \mathcal{I}^{-1}(\mathcal{F}_t)$ and $j \in [r]$, let $\rho_{\mathcal{E}}^{-1}(j) \stackrel{\text{def}}{=} \{x \in \mathcal{E} \mid \rho(x) = j\}$.

Proof of Theorem 2. Let us consider an arbitrary non-empty subset $\mathcal{E} \subseteq \mathcal{I}^{-1}(\mathcal{F}_t)$. Consider a vector $y \in \mathcal{I}(\mathcal{E}) \cap \mathcal{F}_t$ and let y_i be a component of y such that $y_i < u_i$ (such a component always exists since $\mathcal{E}$ is non-empty). Then, by the maximality of y, there exists a vector $x = x^i \in \mathcal{E}$ such that $x \le y + \mathbf{e}^i$, where $\mathbf{e}^i$ is the ith unit vector. Let $j = j_i = \rho(x) \in [r]$ be an index such that x violates the jth inequality of the system.

Claim 1. $y \le \mu^j(x,i)$.

Proof. Let us first note that $x_i = y_i + 1$, since $x_i \le y_i + 1$ and we have $f_j(x) \le t_j$ if $x_i \le y_i$, contradicting the fact that $x \in \mathcal{I}^{-1}(\mathcal{F}_t)$. This means $y_i = \mu^j(x, i)_i$. Moreover, if $x_k < y_k - \alpha_k$ for some $k \ne i$, then we have

$$f_j(y) - f_j(x) = \sum_{h \ne i,k} (f_{hj}(y_h) - f_{hj}(x_h)) + (f_{kj}(y_k) - f_{kj}(x_k)) - (f_{ij}(x_i) - f_{ij}(y_i))$$

$$\ge \Delta_{kj}(x_k + \alpha_k) - \Delta_{ij}(x_i - 1), \tag{14}$$

where the last inequality follows from the monotonicity of the functions f_{ij}, and the facts that $x_k \le y_k$ for all $k \ne i$, $y_i = x_i - 1$, and $y_k \ge x_k + \alpha_k + 1$. Since $\Delta_{kj}(x_k + \alpha_k) - \Delta_{ij}(x_i - 1) \ge 0$ by the definition of $\alpha_k = \alpha_k(x, k, j)$, we get $f_j(y) \ge f_j(x) > t_j$, a contradiction to the fact that $y \in \mathcal{F}_t$. Therefore, $y_k \le x_k + \alpha_k$ must hold for all components $k \ne i$, proving the claim. □

Claim 2. *$y_k = \mu^j(x, i)_k$ for all components $k \in [n]$ for which*

$$\Delta_{kj}(y_k) \ge \Delta_{ij}(y_i). \tag{15}$$

Proof. Let $k \ne i$ satisfy (15), then for $s = 0, 1, \ldots, \alpha_k - 1$, we have

$$\Delta_{kj}(y_k) \ge \Delta_{ij}(y_i) = \Delta_{ij}(x_i - 1) > \Delta_{kj}(x_k + s), \tag{16}$$

by definition of $\alpha_k = \alpha_k(x, i, j)$. Since $x_k \le y_k$, it follows from (16) that $y_k \ge x_k + \alpha_k = \mu^j(x, i)_k$, and therefore the result follows from Claim 1. □

Claim 1 implies that

$$y = \bigwedge_{i \in [n]:\, y_i < u_i} \mu^{j_i}(x^i, i), \tag{17}$$

where for vectors $v, u \in \mathcal{C}$ we let, as before, $v \wedge u$ denote the component-wise minimum of v and u.

Not all of the vectors $\mu^{j_i}(x^i, i)$ are necessary for this representation. Suppose that there exist two vectors $x^i, x^k \in \mathcal{E}$ such that $x^i \le y + \mathbf{e}^i$, $x^k \le y + \mathbf{e}^k$, and $\rho(x^i) = \rho(x^k) = j$. Suppose further that $\Delta_{kj}(x^k_k - 1) \ge \Delta_{ij}(x^i_i - 1)$. Then Claim 2 implies that (17) remains valid even if we drop $\mu^{j_k}(x^k, k)$. In other words, we can identify, for each $j \in [r]$, a single vector $x^{i_j} \in \rho_{\mathcal{E}}^{-1}(j)$, and obtain consequently at most r vectors $\mu^j(x^{i_j}, i_j)$ such that

$$y = \bigwedge_{j \in [r]} \nu^j, \tag{18}$$

where ν^j is either $\mu^j(x^{i_j}, i_j)$ or u. The latter representation readily implies (5). □

For $\mathcal{E} \subseteq \mathcal{C}$, denote by $\mathcal{E}^+ = \{y \in \mathcal{C} \mid y \ge x,\ \text{for some } x \in \mathcal{E}\}$ and $\mathcal{E}^- = \{y \in \mathcal{C} \mid y \le x,\ \text{for some } x \in \mathcal{E}\}$. To prove Theorem 3, we first need the following lemma.

Lemma 2. *Let $\mathcal{F}_t$ be the set of maximal feasible solutions for (3), and let $\mathcal{Y} \subseteq \mathcal{F}_t$ and $\mathcal{X} \subseteq \mathcal{I}^{-1}(\mathcal{F}_t)$ such that $\mathcal{X} \ne \emptyset$. Then $\mathcal{Y} = \mathcal{F}_t$ and $\mathcal{X} = \mathcal{I}^{-1}(\mathcal{F}_t)$ if and only if*

(i) *For all $x \in \mathcal{X}$ and $i \in [n]$ such that $x_i > l_i$, and for all $k \ne i$ such that $\mu^j(x, i)_k < u_k$, where $j = \rho(x)$, the vector $\overline{x} = \overline{x}(x, i, k)$ given by*

$$\overline{x}_h = \begin{cases} x_h - 1 & \text{if } h = i \\ \mu^j(x, i)_h + 1 & \text{if } h = k \\ x_h & \text{otherwise,} \end{cases} \tag{19}$$

is in $\mathcal{X}^+$.

(ii) *For every collection $(x^j \in \rho_{\mathcal{X}}^{-1}(j) \mid j \in [r])$, and for every selection of indices $(k_1, \ldots, k_r)$ such that $x^j_{k_j} > l_{k_j}$, the vector $y = \wedge_{j \in [r]} \nu^j$ is in $\mathcal{X}^+ \cup \mathcal{Y}^-$, where ν^j is either $\mu^j(x^j, k_j)$ or u. (We set $\nu^j = u$ if $\rho_{\mathcal{X}}^{-1}(j) = \emptyset$.)*

Proof. Note that if $x \in \mathcal{X}$, $i, k \in [n]$ and $j \in [r]$ satisfy the conditions specified in (i), and $\overline{x} = \overline{x}(x, i, k)$ is given by (19), then $f_j(\overline{x}) - f_j(x) \ge 0$ follows, implying that both (i) and (ii) are indeed necessary conditions for the duality (i.e., for $\mathcal{Y} = \mathcal{F}_t$ and $\mathcal{X} = \mathcal{I}^{-1}(\mathcal{F}_t)$).

To see the sufficiency, suppose that (i) and (ii) hold, and let y be a maximal element in $\mathcal{C} \setminus (\mathcal{X}^+ \cup \mathcal{Y}^-)$. Since $y \neq u$ by assumption, there is an $i \in [n]$ such that $y_i < u_i$. By the maximality of y, there exists an $x \in \mathcal{X}$ such that $x \leq y + \mathbf{e}^i$. Let $j = \rho(x)$. If $y_k \geq \mu^j(x,i)_k + 1$, for some $k \neq i$, then $y \geq \overline{x}(x,i,k)$, and hence by (i), $y \in \mathcal{X}^+$, yielding a contradiction. We conclude therefore that $y \leq \mu^j(x,i)$, and consequently, as in the proof of Theorem 2, y is in the form given in (18). But then, by (ii), $y \in \mathcal{X}^+ \cup \mathcal{Y}^-$, another contradiction. $\square$

Proof of Theorem 3. Clearly, a vector $x \in \mathcal{I}^{-1}(\mathcal{F}_t)$ can be generated in at most $n \log(\|u-l\|_\infty+1)$ evaluations of the system (3), using binary search. Thus we can assume that we are given two subsets $\mathcal{Y} \subseteq \mathcal{F}_t$ and $\emptyset \neq \mathcal{X} \subseteq \mathcal{I}^{-1}(\mathcal{F}_t)$. We can also assume that

$$\mathcal{Y} \subseteq \mathcal{I}(\mathcal{X}) \quad \text{and} \quad \mathcal{X} \subseteq \mathcal{I}^{-1}(\mathcal{Y}). \tag{20}$$

Indeed, if there is, say, a $y \in \mathcal{Y} \setminus \mathcal{I}(\mathcal{X})$, then let $i \in [n]$ be such that $y + \mathbf{e}^i \notin \mathcal{X}^+$, and find a new minimal vector $x \in \mathcal{I}^{-1}(\mathcal{F}_t) \setminus \mathcal{X}$ by performing at most $n \log(\|u - l\|_\infty + 1)$ evaluations of the system (3). Note that for constant r, (20) together with Theorems 1 and 2 implies that the sizes of $\mathcal{X}$ and $\mathcal{Y}$ are polynomially related: $|\mathcal{X}| \leq rn|\mathcal{Y}|$, $|\mathcal{Y}| \leq (n|\mathcal{X}|)^r$. Consequently, it is enough to show that, given $\mathcal{X}$ and $\mathcal{Y}$, we can generate a new point in $\mathcal{C} \setminus \mathcal{X}^+ \cup \mathcal{Y}^-$ in polynomial time. This can be done using Lemma 2 as follows. In order to compute a new point in $\mathcal{C} \setminus \mathcal{X}^+ \cup \mathcal{Y}^-$, we may assume that each chain $\mathcal{C}_i$ is composed of only those elements that appear in $\mathcal{X}$ and $\mathcal{Y}$:

$$\{l_i, u_i\} \cup \{x_i \,:\, x \in \mathcal{X}\} \cup \{y_i \,:\, y \in \mathcal{Y}\} \tag{21}$$

for $i = 1, \ldots, n$. It follows from (20) that the set above contains all the ith components of the predecessors of $\mathcal{X}$ and the successors of $\mathcal{Y}$, i.e. $\{x_i - 1 \,:\, x \in \mathcal{X},\ x_i \neq l_i\} \cup \{y_i + 1 \,:\, y \in \mathcal{Y},\ y_i \neq u_i\}$. To see the validity of the assumption, let $\{p_i^0, p_i^1, \ldots, p_i^{k_i}\}$ be the set specified in (21), where $p_i^0 < p_i^1 < \cdots < p_i^{k_i}$, and let $\mathcal{C}'_i = \{0, 1, \ldots, k_i\}$ for $i = 1, \ldots, n$. Define the functions $f'_{ij} : \mathcal{C}'_i \longrightarrow \mathbb{R}$, for $i = 1, \ldots, n$ and $j = 1, \ldots, r$ by $f'_{ij}(k) = f_{ij}(p_i^k)$ for $k \in \mathcal{C}'_i$. Let $\mathcal{X}'$ and $\mathcal{Y}'$ be the sets of elements of $\mathcal{C}'$ corresponding respectively to elements of $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{C}$. Clearly, the functions f'_{ij} are monotone, and $\mathcal{Y}' \subseteq \mathcal{F}'_t$, $\mathcal{X}' \subseteq \mathcal{I}^{-1}(\mathcal{F}'_t)$, where $t = (t_1, \ldots, t_r) \in \mathbb{R}^r$ and $\mathcal{F}'_t$ is the family of maximal feasible solutions of the system

$$\sum_{i=1}^{n} f'_{ij}(x_i) \leq t_j, \quad j = 1, \ldots, r,$$

and the operator $\mathcal{I}^{-1}(\cdot)$ is computed with respect to $\mathcal{C}'$. Moreover, we have $\mathcal{X}^+ \cup \mathcal{Y}^- = \mathcal{C}$ if and only if $(\mathcal{X}')^+ \cup (\mathcal{Y}')^- = \mathcal{C}'$. [If y is maximal in $\mathcal{C} \setminus (\mathcal{X}^+ \cup \mathcal{Y}^-)$, $i \in [n]$, and $y_i \neq u_i$ then $y + \mathbf{e}^i \in \mathcal{X}^+$ and therefore there is an $x \in \mathcal{X}$ such that $y \geq x - \mathbf{e}^i$. But then y_i must be equal to $x_i - 1$, i.e. y_i maps to a point in $\mathcal{C}'_i$.]

It follows then that computing the vector $\mu^j(x,i)$ for given $x \in \mathcal{X}$, $i \in [n]$ and $j \in [r]$ can be carried out in time polynomial in n and $|\mathcal{X}| + |\mathcal{Y}|$. Thus, we can compute the set of vectors given by (19) and check if each belongs to $\mathcal{X}^+$. If not, we obtain a new element in $\mathcal{C} \setminus (\mathcal{X}^+ \cup \mathcal{Y}^-)$, which can be extended to an element in $\mathcal{I}^{-1}(\mathcal{F}_t) \setminus \mathcal{X}$ in time polynomial in n, r and $\log(\|u - l\|_\infty + 1)$. Otherwise, we perform the check in part (ii) of the lemma which either gives us a new point $z \in \mathcal{C} \setminus (\mathcal{X}^+ \cup \mathcal{Y}^-)$ (which can be extended to either an element $y \in \mathcal{F}_t \setminus \mathcal{Y}$ or $x \in \mathcal{I}^{-1}(\mathcal{F}_t) \setminus \mathcal{X}$, depending on whether z is feasible or infeasible for the system (3)), or proves that the current sets $\mathcal{X}$ and $\mathcal{Y}$ are complete, in which case we have obtained all the required elements, i.e., $\mathcal{Y} = \mathcal{F}_t$ and $\mathcal{X} = \mathcal{I}^{-1}(\mathcal{F}_t)$. $\square$

7. Generalizations

In this section, we give some generalizations of the intersection lemma and discuss some further applications.

7.1. Intersection lemma for meet semi-lattices

Let $\mathcal{P}_i$, $i = 1, \ldots, n$ be given finite partial orders such that for any index i and any two elements $x, y \in \mathcal{P}_i$, elements x and y have a *unique minimum*, i.e. the *meet* $x \wedge y \overset{\text{def}}{=} \min(x,y) \in \mathcal{P}_i$ exists and is well defined. Denote by "$\preceq$" the precedence relation on $\mathcal{P}$, and for $\mathcal{E} \subseteq \mathcal{P}$, let $\mathcal{E}^+ = \{y \in \mathcal{P} \mid y \succeq x \text{ for some } x \in \mathcal{E}\}$ and

$\mathcal{E}^- = \{y \in \mathcal{P} \mid y \preceq x \text{ for some } x \in \mathcal{E}\}$. For simplicity, we write x^+ and x^- instead of $\{x\}^+$ and $\{x\}^-$, respectively. For $i \in [n]$ and $x \in \mathcal{P}_i$, define

$$q_i(x) = |\{z \in \mathcal{P}_i \ : \ z \notin x^- \text{ and } z \text{ has an immediate predecessor } z' \preceq x\}|,$$

and let

$$q(y) \stackrel{\text{def}}{=} \sum_{i=1}^{n} q_i(y_i) \tag{22}$$

for $y \in \mathcal{P} \stackrel{\text{def}}{=} \mathcal{P}_1 \times \cdots \times \mathcal{P}_n$. Note that this definition of $q(y)$ coincides with the one given in (8), if each $\mathcal{P}_i$ is a total order.

Lemma 3. *Let $\mathcal{P}_i$, $i = 1, \ldots, n$, be given finite meet-semi lattices, let $w : \cup_{i=1}^n \mathcal{P}_i \longrightarrow \mathbb{R}_+$ be a function assigning a non-negative weight to each element in $\cup_{i=1}^n \mathcal{P}_i$, and let $t \in \mathbb{R}_+$ be a given positive threshold. Assume that $\mathcal{X}$ and $\mathcal{Y} \neq \emptyset$ are subsets of $\mathcal{P} = \mathcal{P}_1 \times \cdots \times \mathcal{P}_n$ such that*

(i) *for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ we have $\sum_{i=1}^n w(x_i^-) > t \geq \sum_{i=1}^n w(y_i^-)$, where $w(\mathcal{Q}) \stackrel{\text{def}}{=} \sum_{z \in \mathcal{Q}} w(z)$, for $\mathcal{Q} \subseteq \mathcal{P}_i$ and $i \in [n]$;*
(ii) *For every $x' \neq x'' \in \mathcal{X}$ there exists a $y \in \mathcal{Y}$ such that $y \succeq x' \wedge x''$.*

Then we have

$$|\mathcal{X}| \leq \sum_{y \in \mathcal{Y}} q(y). \tag{23}$$

In particular, $|\mathcal{X}| \leq (\sum_{i=1}^n |\mathcal{P}_i| - n)|\mathcal{Y}|$.

Proof. We may assume that $\mathcal{Y}$ is a minimal family for the above properties. Clearly, for $|\mathcal{X}| \leq 1$ the statement is true since $\mathcal{Y}$ is non-empty and $q(y) = 0$ for $y \in \mathcal{Y}$ implies by (i) that $\mathcal{X} = \emptyset$. We shall prove the lemma by induction on $|\mathcal{X}| \geq 1$.

As in the proof of Lemma 1, let us define for every $i = 1, \ldots, n$ and $z \in \mathcal{P}_i$ the families

$$\mathcal{X}(i, z) = \{x \in \mathcal{X} : x_i \succeq z\}, \qquad \mathcal{Y}(i, z) = \{y \in \mathcal{Y} \ : \ y_i \succeq z\}.$$

For $i = 1, \ldots, n$, let

$$\mathcal{Z}_i \stackrel{\text{def}}{=} \{\text{minimal } z \in \mathcal{P}_i \ : \ |\mathcal{X}(i, z)| = 1 \text{ and } |\mathcal{Y}(i, z)| = 0\},$$

and let $\mathcal{P}_i' \stackrel{\text{def}}{=} \mathcal{P}_i \setminus \mathcal{Z}_i^+$, $\mathcal{X}_i \stackrel{\text{def}}{=} \{x \in \mathcal{X} \mid x_i \in \mathcal{Z}_i^+\}$ and $\mathcal{X}' \stackrel{\text{def}}{=} \mathcal{X} \setminus (\bigcup_{i=1}^n \mathcal{X}_i)$. Let further $q'(y)$ for $y \in \mathcal{Y}$ be the value of (22), computed with respect to $\mathcal{P}' \stackrel{\text{def}}{=} \mathcal{P}_1' \times \cdots \times \mathcal{P}_n'$. Note that

(a) $\mathcal{P}_i'$ is a meet-semi lattice, for $i = 1, \ldots, n$.
(b) For $i = 1, \ldots, n$, the minimality of $z \in \mathcal{Z}_i$ implies that z has an immediate predecessor z' with $z' \preceq y_i$ for some $y \in \mathcal{Y}$ and hence $\sum_{y \in \mathcal{Y}} q'(y) \leq \sum_{y \in \mathcal{Y}} q(y) - \sum_{i=1}^n |\mathcal{Z}_i|$. We furthermore have $|\mathcal{X}'| \geq |\mathcal{X}| - \sum_{i=1}^n |\mathcal{Z}_i|$ since $|\mathcal{X}(i, z)| = 1$ for all $z \in \mathcal{Z}_i$, by definition.
(c) For $i = 1, \ldots, n$, the definition of $\mathcal{P}_i'$ and (ii) imply that $\mathcal{Y}(i, z) \neq \emptyset$ whenever $\mathcal{X}'(i, z) \neq \emptyset$, for all $i = 1, \ldots, n$, and for all $z \in \mathcal{P}_i'$.

Thus, the families $\mathcal{X}'(i, z)$ and $\mathcal{Y}(i, z)$ satisfy the conditions (i) and (ii) of the statement with respect to the partial order $\mathcal{P}'$ and we can therefore assume by induction on the number of elements in $\mathcal{X}$ that

$$|\mathcal{X}'(i, z)| \leq \sum_{y \in \mathcal{Y}(i,z)} q'(y) \tag{24}$$

whenever

$$|\mathcal{X}'(i, z)| < |\mathcal{X}'|.$$

Let us note next that for every index i we can assume that $\mathcal{X}'(i,z)=\mathcal{X}'$ for at most one value $z\in\mathcal{P}'_i$. Clearly, $z=\wedge\{x_i \mid x\in\mathcal{X}'\}$ is such a value, and $z^-\subseteq\mathcal{P}'_i$ is exactly the subset of all such values (due to the existence of a unique minimum). Thus $z^-\subseteq x_i^-$ for all $x\in\mathcal{X}'$, and due to the minimality of $\mathcal{Y}$, we also have $z^-\subseteq y_i^-$ for all $y\in\mathcal{Y}$. Hence, redefining the partial order $\mathcal{P}'_i$, by deleting all elements $z'\not\succeq z$ from it, yields a new partial order in which there is still a unique minimum for every two different elements and which will not change the sets $\mathcal{X}'$ and $\mathcal{Y}$. Furthermore, for every element $z'\in\mathcal{P}'_i$ with $z'\succeq z$, if we replace the weight $w(z')$ by the sum

$$\sum_{z''\in\mathcal{P}'_i:\ z'=z\vee z''} w(z''),$$

where $z\vee z'\stackrel{\text{def}}{=}\max(z,z')$ is uniquely defined if it exists, then clearly conditions (i) and (ii) of the statement remain valid with respect to the new partial order and weights. We can assume therefore without loss of generality that $\mathcal{X}'(i,z)=\mathcal{X}'$ only at $z=l_i$, the minimum element of $\mathcal{P}'_i$, for $i=1,\ldots,n$.

Let us then multiply each inequality (24) by the non-negative weight $w(z)$ and sum up the resulting inequalities, for all indices i and for all values $z\neq l_i$ (for which $|\mathcal{X}'(i,z)|\neq|\mathcal{X}'|$), yielding

$$\sum_{i=1}^{n}\sum_{z\neq l_i} w(z)|\mathcal{X}'(i,z)| \le \sum_{i=1}^{n}\sum_{z\neq l_i} w(z)\sum_{y\in\mathcal{Y}(i,z)} q'(y). \tag{25}$$

The left hand side of (25) is equal to

$$L=\sum_{x\in\mathcal{X}'}\sum_{i=1}^{n} w(x_i^-)-\sum_{i=1}^{n} w(l_i)|\mathcal{X}'|$$

and the right hand side is

$$R=\sum_{y\in\mathcal{Y}} q'(y)\left(\sum_{i=1}^{n} w(y_i^-)-\sum_{i=1}^{n} w(l_i)\right).$$

Thus, on the one hand we get by (i) and (b) that

$$\left(t-\sum_{i=1}^{n} w(l_i)\right)\left(|\mathcal{X}|-\sum_{i=1}^{n}|\mathcal{Z}_i|\right)\le\left(t-\sum_{i=1}^{n} w(l_i)\right)|\mathcal{X}'|<L \tag{26}$$

and on the other hand, again by (i) and (b), we obtain

$$R\le\left(t-\sum_{i=1}^{n} w(l_i)\right)\sum_{y\in\mathcal{Y}} q'(y)\le\left(t-\sum_{i=1}^{n} w(l_i)\right)\left(\sum_{y\in\mathcal{Y}} q(y)-\sum_{i=1}^{n}|\mathcal{Z}_i|\right). \tag{27}$$

If $t<\sum_{i=1}^{n} w(l_i)$ then we get a contradiction to the assumption $|\mathcal{Y}|\ge 1$. If $t=\sum_{i=1}^{n} w(l_i)$, we can replace t by $\min\{\sum_{i=1}^{n} w(x_i^-)\mid x\in\mathcal{X}\}-\epsilon$, for a sufficiently small $\epsilon>0$, and assume therefore that $t>\sum_{i=1}^{n} w(l_i)$. Therefore (23) follows from (25), (26) and (27). □

The bound of Lemma 3 is best possible as illustrated by the following example. Let each $\mathcal{P}_i=\{l_i,r_i^1,\ldots,r_i^{k_i}\}$ with minimum element l_i and relations $l_i\prec r_i^j$ for $j=1,\ldots,k_i$. Let $\mathcal{X}=\{(l_1,\ldots,l_{i-1},r_i^j,l_{i+1},\ldots,l_n)\ :\ j=1,\ldots,k_i,\ \ i=1,\ldots,n\}$, and let $y=(l_1,\ldots,l_n)$ be the only element of $\mathcal{Y}$. Then for the set of weights $w(l_i)=\epsilon$, $w(r_i^j)=1$, for $j=1,\ldots,k_i$, $i=1,\ldots,n$, and for $t=n\epsilon$ for some $\epsilon>0$, we have $|\mathcal{X}|=\sum_{i=1}^{n} k_i=\sum_{i=1}^{n} q_i(y_i)$.

Note also that Lemma 1 can be derived as a special case of Lemma 3. Indeed, given $\mathcal{X},\mathcal{Y}\subseteq\mathbb{R}^n$, let $\{p_i : p\in\mathcal{X}\cup\mathcal{Y}\}\stackrel{\text{def}}{=}\{p_i^0,p_i^1,\ldots,p_i^{k_i}\}$, where $p_i^0<p_i^1<\cdots<p_i^{k_i}$, and define $\mathcal{P}_i$ to be the chain $\{0,1,\ldots,k_i\}$, for $i=1,\ldots,n$. We may assume without loss of generality that $p^0=(p_1^0,\ldots,p_n^0)=0$ and $w_i=1$ for all i, since we can translate the point sets $\mathcal{X}$ and $\mathcal{Y}$ without violating properties (P1) and (P2). Define the non-negative weights $w(p_i^0)=p_i^0$ and $w(p_i^j)=p_i^j-p_i^{j-1}$ for $j=1,\ldots,k_i$ and $i=1\ldots,n$. Now Lemma 1 becomes a consequence of (23).

7.2. r-Intersection lemma

Lemma 3 can be further generalized as follows. Given two finite sets of elements $\mathcal{X}$ and $\mathcal{Y}$ in the product $\mathcal{P} \stackrel{\text{def}}{=} \mathcal{P}_1 \times \cdots \times \mathcal{P}_n$ of n meet semi-lattices, and an integer $r \geq 2$, consider the following property :

(ii′) For any r distinct elements $x^1, x^2, \ldots, x^r \in \mathcal{X}$, their componentwise meet $x^1 \wedge x^2 \wedge \cdots \wedge x^r$ is dominated by some $y \in \mathcal{Y}$, i.e. $x^1 \wedge x^2 \wedge \cdots \wedge x^r \preceq y$.

Lemma 4. *If $\mathcal{X}$ and $\mathcal{Y} \neq \emptyset$ are two finite sets of elements in $\mathcal{P}$ satisfying properties* (i) *of Lemma* 3 *and* (ii′) *above, then*

$$|\mathcal{X}| \leq (r-1) \sum_{y \in \mathcal{Y}} q(y).$$

The proof of this lemma is a straightforward modification of that of Lemma 3.

7.3. Systems of monotone inequalities on sums of separable functions with bounded number of variables

We shall consider in this section *multi-hypergraphs*, i.e. hypergraphs $\mathcal{H} \subseteq 2^{[n]}$ in which every hyperedge has an integral multiplicity. For instance, if we indicate multiplicities in parentheses, then $\mathcal{H} = \{H_1 = \{1,2\}(1),\ H_2 = \{1,2\}(2),\ H_3 = \{3\}(1)\}$ is a multi-hypergraph consisting of three hyperedges of multiplicities 1, 2, and 1, respectively. Let us define $\dim(\mathcal{H}) = \max\{|H| \,:\, H \in \mathcal{H}\}$. For instance, $\dim(\mathcal{H}) = 2$ for the above example, since hyperedges H_1 and H_2 both have two elements, while hyperedge H_3 has only one.

We can generalize Theorem 1 as follows. Let $\mathcal{H}_1, \ldots, \mathcal{H}_r \subseteq 2^{[n]}$ be r multi-hypergraphs on n vertices, and let $\mathcal{C} = \mathcal{C}_1 \times \cdots \times \mathcal{C}_n = \{x \in \mathbb{R}^n \mid l \leq x \leq u\}$, where $l, u \in \mathbb{R}^n$. For $j = 1, \ldots, r$, $H \in \mathcal{H}_j$, and $i \in H$, let $f_{H,i,j} : \mathcal{C}_i \longrightarrow \mathbb{R}_+$ be a single-variable (polynomial-time computable) monotone function. Consider a system of r inequalities

$$\sum_{H \in \mathcal{H}_j} \prod_{i \in H} f_{H,i,j}(x_i) \leq t_j, \quad j = 1, \ldots, r, \tag{28}$$

over $x \in \mathcal{C}$, where $t_1, \ldots, t_r$ are given real thresholds.

For instance, if $r = 1$, $\mathcal{H}_1 = \mathcal{H}$ is the multi-hypergraph considered in the example above, and $f_{H_1,1,1}(x_1) = x_1^3$, $f_{H_2,1,1}(x_1) = x_1$, $f_{H_1,2,1}(x_2) = f_{H_2,2,1}(x_2) = x_2$, and $f_{H_3,3,1}(x_3) = x_3^5$, then (28) consists of the following single inequality:

$$x_1^3 x_2 + 2x_1 x_2 + x_3^5 \leq t_1.$$

Theorem 6. *If $\dim(\mathcal{H}_j) \leq const$ for all $j = 1, \ldots, r$, then all maximal feasible solutions of a system* (28) *can be generated in incremental quasi-polynomial time.*

Theorem 6 is an immediate consequence of the following statement:

Theorem 7. *Let $\mathcal{H}_1, \ldots, \mathcal{H}_r \subseteq 2^{[n]}$ be r multi-hypergraphs on n vertices. For $j = 1, \ldots, r$, $H \in \mathcal{H}_j$, and $i \in H$, let $f_{H,i,j} : \mathcal{C}_i \longrightarrow \mathbb{R}_+$ be a single-variable monotone function. If $\mathcal{F} \subseteq \mathcal{C}$ is the family of all maximal feasible solutions of* (28)*, and $\mathcal{E} \subseteq \mathcal{F}$ is non-empty, then*

$$\left|\mathcal{I}^{-1}(\mathcal{E}) \cap \mathcal{I}^{-1}(\mathcal{F})\right| \leq \left(\sum_{j=1}^{r} \sum_{H \in \mathcal{H}_j} |H|(2|\mathcal{E}|+1)^{|H|-1}\right) |\mathcal{E}|.$$

In particular,

$$\left|\mathcal{I}^{-1}(\mathcal{E}) \cap \mathcal{I}^{-1}(\mathcal{F})\right| \leq d \left(\sum_{j=1}^{r} |\mathcal{H}_j|\right) (2|\mathcal{E}|+1)^{d-1} |\mathcal{E}|,$$

where $d = \max\{\dim(\mathcal{H}_1), \ldots, \dim(\mathcal{H}_r)\}$.

Proof. We may assume without loss of generality that each chain $\mathcal{C}_i$ is composed only of those elements that appear in $\mathcal{E}$ and their successors:

$$\mathcal{C}_i = \{l_i\} \cup \{y_i :\ y \in \mathcal{E}\} \cup \{y_i + 1 :\ y \in \mathcal{E},\ y_i \neq u_i\},$$

for $i = 1, \ldots, n$. [If $x \in \mathcal{I}^{-1}(\mathcal{E})$, $i \in [n]$, and $x_i \neq l_i$ then $x - \mathbf{e}^i \in \mathcal{E}^-$ and therefore there is a $y \in \mathcal{E}$ such that $y \geq x - \mathbf{e}^i$. But then y_i must be equal to $x_i - 1$, i.e. $x_i = y_i + 1 \in \mathcal{C}_i$.] Assume also without loss of generality that $r = 1$ and let $\mathcal{H} \subseteq 2^{[n]}$ and $f(x) = \sum_{H \in \mathcal{H}} \prod_{i \in H} f_{H,i}(x_i)$. Given $t \in \mathbb{R}_+$, and a non-empty subset $\mathcal{E}$ of the maximal feasible solutions of the inequality $f(x) \leq t$, let $\mathcal{X} = \mathcal{I}^{-1}(\mathcal{E}) \cap \mathcal{I}^{-1}(\mathcal{F})$. We use Lemma 3 to prove the theorem. Define the partial orders

$$\mathcal{P}_H = \bigotimes_{i \in H} \mathcal{C}_i, \quad \text{for } H \in \mathcal{H} \quad \text{and} \quad \mathcal{P} = \bigotimes_{H \in \mathcal{H}} \mathcal{P}_H.$$

For an element $z = (x_i \in \mathcal{C}_i : i \in H) \in \mathcal{P}_H$, let us associate the non-negative weight $w(z) = \prod_{i \in H}(f_{H,i}(x_i) - f_{H,i}(x_i - 1))$, where we assume that $f_{H,i}(l_i - 1) \stackrel{\text{def}}{=} 0$ for all $H \in \mathcal{H}$ and $i \in H$. Consider the monotone mapping $\phi : \mathcal{C} \longrightarrow \mathcal{P}$ defined by: $\phi(x) = ((x_i \ :\ i \in H) \ :\ H \in \mathcal{H})$ for $x \in \mathcal{C}$, and let $\mathcal{X}' = \{\phi(x) \mid x \in \mathcal{X}\}$, and $\mathcal{Y}' = \{\phi(y) \mid y \in \mathcal{E}\}$. Note that for any $H \in \mathcal{H}$ and $y \in \mathcal{P}_H$ we have

$$|q_H(y)| \leq \sum_{i \in H} \prod_{j \in H, j \neq i} |\mathcal{C}_j| \leq |H|(2|\mathcal{E}| + 1)^{|H|-1}.$$

Thus with respect to the above weights and the partial order $\mathcal{P}$, the families $\mathcal{X}'$ and $\mathcal{Y}'$ satisfy conditions (i) and (ii) of Lemma 3, and consequently

$$|\mathcal{X}| = |\mathcal{X}'| \leq \sum_{y \in \mathcal{Y}'} q_H(y) \leq \sum_{H \in \mathcal{H}} |H|(2|\mathcal{E}| + 1)^{|H|-1}|\mathcal{E}|.$$

The theorem follows. □

On the negative side, we have the following proposition.

Proposition 3. *Given a hypergraph $\mathcal{H} \subseteq 2^{[n]}$ and an integer threshold t, incrementally generating all minimal infeasible vectors for the inequality $f(x) = \sum_{H \in \mathcal{H}} \prod_{i \in H} x_i \leq t$ over $x \in \{0, 1\}^{[n]}$ is NP-hard, even if* $\dim(\mathcal{H}) = 2$.

Proof. Again, we reduce the problem from the following well-known NP-complete problem: Given a graph $G = (V, E)$ and an integer t, determine if G contains an independent set of size at least t. To do this let us associate a *binary* variable x_i with each vertex $i \in V$, and define the monotone function

$$f(x) = (t - 2) \cdot \sum_{\{i,j\} \in E} x_i x_j + \sum_{i \in V} x_i,$$

over the elements $x \in \{0, 1\}^V$. Let $\mathcal{Y} \subseteq \{0, 1\}^V$ be the set of incidence vectors of the edges of G. Then $\mathcal{Y}$ is a subset of the minimal infeasible vectors for the inequality $f(x) \leq t - 1$, and it is easy to see that there are no other minimal infeasible vectors if an only if there is no independent set of G of size at least t. □

7.4. Maximal packings/coverings of points into/by boxes

Let $\mathcal{S}$ be a set of points in $\mathbb{R}^n$. Let $C : \mathcal{S} \longrightarrow \{1, 2, \ldots, r\}$ and $w : \mathcal{S} \longrightarrow \mathbb{R}_+$ be respectively a coloring and a weighting of the point set $\mathcal{S}$, i.e. mappings that assign respectively one of r colors and a non-negative real weight to each point in $\mathcal{S}$. Given a non-negative threshold vector $t = (t_1, \ldots, t_r) \in \mathbb{R}^r_+$, let us define a *packing* of the point set $\mathcal{S}$, with respect to (C, w, t), to be a box containing (in its interior) a subset of $\mathcal{S}_i \stackrel{\text{def}}{=} \{p \in \mathcal{S} \mid C(p) = i\}$ of total weight at most t_i for all $i = 1, \ldots, r$. Let us define conversely a (C, w, t)*-covering* of $\mathcal{S}$, to be any box that contains a subset of $\mathcal{S}_i$ of total weight greater than t_i for some $i = 1, \ldots, r$. Denote respectively by $\mathcal{F}_{\mathcal{S},C,w,t}$ and $\mathcal{I}^{-1}(\mathcal{F}_{\mathcal{S},C,w,t})$ the families of all maximal packings and all minimal coverings of the point set $\mathcal{S}$ with respect to (C, w, t). Clearly, if $r = 1$, $t = k$, and all weights are ones, then $\mathcal{F}_{\mathcal{S},C,w,t}$ is just the family of maximal k-boxes discussed in Section 4. Therefore, Theorem 5 is a special case of the following.

Theorem 8. *All maximal packings of a given point set $\mathcal{S} \subseteq \mathbb{R}^n$, with respect to a given coloring $C : \mathcal{S} \longrightarrow \{1, 2, \ldots, r\}$, a non-negative weight $w : \mathcal{S} \longrightarrow \mathbb{R}_+$, and a given threshold vector $t \in \mathbb{R}^r_+$, can be generated in incremental quasi polynomial time.*

This follows again from a generalization of the dual-bounding inequality (7), which can be proved using the intersection lemma:

Theorem 9. *Let $\mathcal{S}$ be a given set of points in $\mathbb{R}^n$, $C : \mathcal{S} \longrightarrow \{1, 2, \ldots, r\}$ and $w : \mathcal{S} \longrightarrow \mathbb{R}_+$ be respectively a coloring and a weighting of $\mathcal{S}$, and $t \in \mathbb{R}^r_+$ be a given non-negative real-threshold. If $\mathcal{F} = \mathcal{F}_{\mathcal{S},C,w,t}$ is the set of packings of the point set $\mathcal{S}$, with respect to (C, w, t), then*

$$|\mathcal{I}^{-1}(\mathcal{Y}) \cap \mathcal{I}^{-1}(\mathcal{F})| \leq \sum_{i=1}^{r} \sum_{y \in \mathcal{Y}} |\{p \in \mathcal{S}_i \mid \textit{point } p \notin \textit{the interior of box } y\}|, \tag{29}$$

for any $\emptyset \neq \mathcal{Y} \subseteq \mathcal{F}$, where $\mathcal{S}_i = \{p \in \mathcal{S} \mid C(p) = i\}$. In particular, $|\mathcal{I}^{-1}(\mathcal{F})| \leq |\mathcal{S}||\mathcal{F}|$.

7.5. Maximal packings with certain geometric properties

We conclude with one more application of Lemma 3. Let $\mathcal{S}$ be a set of points in $\mathbb{R}^n$. For $i = 1, \ldots, n$, consider the set of projection points $\mathcal{P}_i \stackrel{\text{def}}{=} \{p_i \in \mathbb{R} \mid p \in \mathcal{S}\}$, and let $\mathcal{L}_i$ be the *lattice of intervals* whose elements are the different intervals defined by the projection points $\mathcal{P}_i$, and ordered by containment "$\succeq$". The meet of any two intervals in $\mathcal{L}_i$ is their *intersection*, and the join is their *span*, i.e. the minimum interval containing both of them. The minimum element l_i of $\mathcal{L}_i$ is the empty interval. Let $\mathcal{L} = \mathcal{L}_1 \times \cdots \times \mathcal{L}_n$, and for a box $x \in \mathcal{L}$, and $i \in [n]$, denote by $|x_i|$ the length of the interval x_i. Let $f_{ij} : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, $i = 1, 2, \ldots, n$, $j = 1, \ldots, r$ be monotone real functions for which $f_{ij}(|x|)$ is *supermodular* over $x \in \mathcal{L}$, i.e., for which we have $f_{ij}(|x|) \geq f_{ij}(|y|)$ for $x \succeq y$, and

$$f_{ij}(|x \vee y|) + f_{ij}(|x \wedge y|) \geq f_{ij}(|x|) + f_{ij}(|y|) \tag{30}$$

for all $x, y \in \mathcal{L}_i$. Let us also say that $f : \mathcal{L}_i \longrightarrow \mathbb{R}_+$ is *locally supermodular* if (30) is satisfied for all $x, y \in \mathcal{L}_i$ for which $x \vee y$ is an immediate successor of x, y. It is not hard to see that local supermodularity is equivalent with the supermodularity of a monotone function on the lattice $\mathcal{L}_i$ (the same is not true for non-monotone functions).

Consider the system of inequalities

$$\sum_{i=1}^{n} f_{ij}(|x_i|) \leq t_j, \quad j = 1, \ldots, r, \tag{31}$$

over the set of n-dimensional boxes $x \in \mathcal{L}$, where $t = (t_1, \ldots, t_r)$ is a given nonnegative r-dimensional real vector.

Let us denote by $\mathcal{F}_{\mathcal{S},t}$ the set of all maximal feasible solutions for (31).

Theorem 10. *Let $\mathcal{S} \subseteq \mathbb{R}^n$ be a given point set, $f_{ij} : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, $i = 1, 2, \ldots, n$, $j = 1, \ldots, r$ be monotone* supermodular *functions, and $t \in \mathbb{R}^r_+$ be a given threshold vector. Then for any non-empty subset $\mathcal{Y}$ of the maximal feasible solutions $\mathcal{F}_{\mathcal{S},t}$ of (31), we have*

$$|\mathcal{I}^{-1}(\mathcal{Y}) \cap \mathcal{I}^{-1}(\mathcal{F}_{\mathcal{S},t})| \leq rn(2|\mathcal{S}| - 1)|\mathcal{Y}|. \tag{32}$$

Proof. Naturally, the elements in each lattice $\mathcal{L}_i$ can be ranked from 0, at the minimum element l_i, to k_i at the maximum element, where k_i is the number of minimal elements in $\mathcal{L}_i \setminus \{l_i\}$. For $y \in \mathcal{L}_i$, if y has two immediate predecessors in $\mathcal{L}_i$, let us denote them by y' and y''. Now, for $j = 1, \ldots, r$, consider that the following set of weights

$$w^j(y) = \begin{cases} f_{ij}(|y|) & \text{if rank}(y) = 0 \\ f_{ij}(|y|) - f_{ij}(|l_i|) & \text{if rank}(y) = 1 \\ f_{ij}(|y|) - f_{ij}(|y'|) - f_{ij}(|y''|) + f_{ij}(|y' \wedge y''|) & \text{otherwise} \end{cases} \tag{33}$$

defined on the elements $y \in \mathcal{L}_i$, $i = 1, \ldots, n$. (Actually, these are the so-called *Möbius coefficients* of the function f_{ij} on the lattice $\mathcal{L}_i$, see, e.g., [3].)

Then it immediately follows from the monotonicity and supermodularity of the functions f_{ij} that the weights (33) are nonnegative. Furthermore, we have $w^j(x^-) = f_{ij}(|x|)$, for any $x \in \mathcal{L}_i$, and $i \in [n]$. This can be easily seen by induction on the rank of the element $x \in \mathcal{L}_i$. Indeed, the statement is trivially true if $\operatorname{rank}(x) = 0$. If $\operatorname{rank}(x) \geq 1$ then we let $x^- \setminus (x')^- \stackrel{\text{def}}{=} \{x^1, \dots, x^k\}$, where we assume that $x = x^k \succ x^{k-1} = x'' \succ x^{k-2} \succ \cdots \succ x^1$. For $h = 1, \dots, k$, let y^h by the predecessor of x^h in $(x')^-$. Note that $y^1 = l_i$ and $y^k = x'$.

Claim 1. $\sum_{h=1}^k w^j(x^h) = f_{ij}(|x^k|) - f_{ij}(|y^k|)$.

Proof. For $k = 1$, the statement is true by (33) since $\operatorname{rank}(x^k) = 1$. For $k > 1$, we have by induction and definition of the weights w^j

$$\begin{aligned}\sum_{h=1}^{k} w^j(x^h) &= w^j(x^k) + \textstyle\sum_{h=1}^{k-1} w^j(x^h)\\ &= f_{ij}(|x^k|) - f_{ij}(|x^{k-1}|) - f_{ij}(|y^k|) + f_{ij}(|y^{k-1}|) + \sum_{h=1}^{k-1} w^j(x^h)\\ &= f_{ij}(|x|) - f_{ij}(|y^k|). \quad \square\end{aligned}$$

Now we apply induction at x' to get $w^j((x')^-) = f_{ij}(|x'|)$, and thus the above claim gives

$$w^j(x^-) = w^j((x')^-) + \sum_{h=1}^{k} w^j(x^h) = f_{ij}(|x'|) + f_{ij}(|x^k|) - f_{ij}(|y^k|) = f_{ij}(|x|).$$

Now (32) becomes a consequence of Lemma 3 since $q(y) \leq 2|\mathcal{S}| - 1$ for all $y \in \mathcal{L}_i$ and all $i \in [n]$. $\square$

Corollary 3. *Let $\mathcal{S} \subseteq \mathbb{R}^n$ be a given point set, $f_{ij} : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, $i = 1, 2, \dots, n$, $j = 1, \dots, r$ be monotone* convex *functions, and $t \in \mathbb{R}_+^r$ be a given threshold vector. Then for any non-empty subset $\mathcal{Y}$ of the maximal feasible solutions $\mathcal{F}_{\mathcal{S},t}$ of the system*

$$\sum_{i=1}^{n} f_{ij}(|x_i|) \leq t_j, \quad j = 1, \dots, r,$$

we have

$$|\mathcal{I}^{-1}(\mathcal{Y}) \cap \mathcal{I}^{-1}(\mathcal{F}_{\mathcal{S},t})| \leq rn(2|\mathcal{S}| - 1)|\mathcal{Y}|. \tag{34}$$

Proof. By Theorem 10, it is enough to verify that the functions f_{ij} are locally supermodular on the lattice $\mathcal{L}$. For this, consider two elements $y', y'' \in \mathcal{L}$, for which $y = y' \vee y''$ is an immediate successor of both y' and y'', i.e. y is the span of y' and y''. Then

$$\begin{aligned} f_{ij}(|y|) - f_{ij}(|y'|) &= f_{ij}(|y'| + |y''| - |y' \wedge y''|) - f_{ij}(|y'|)\\ &\geq f_{ij}(|y' \wedge y''| + |y''| - |y' \wedge y''|) - f_{ij}(|y' \wedge y''|)\\ &= f_{ij}(|y''|) - f_{ij}(|y' \wedge y''|), \end{aligned} \tag{35}$$

where (35) follows from the convexity and monotonicity of f_{ij}. $\square$

Finally, we mention two applications of Corollary 3:

- Given a set of points $\mathcal{S} \subseteq \mathbb{R}^n$, a coloring $C : \mathcal{S} \longrightarrow \{1, 2, \dots, r\}$, a weighting $w : \mathcal{S} \longrightarrow \mathbb{R}_+$, and a non-negative real threshold $t \in \mathbb{R}_+^r$, generate all maximal (w, C, t)-packings of $\mathcal{S}$ with diameter not exceeding a given threshold $\delta \geq 0$. If $x \in \mathcal{L}$ is such a packing, then it must further satisfy the inequality $(\sum_{i=1}^n |x_i|^p)^{1/p} \leq \delta$ which is in the form covered by Corollary 3 for any finite $p \geq 1$.
- Given n sets $\mathbb{P}_1, \dots, \mathbb{P}_n \subseteq \mathbb{R}$, and a positive real threshold δ, generate all minimal boxes $[a, b] \in \mathcal{L}$ with $\{a_i, b_i\} \subseteq \mathbb{P}_i$, for $i = 1, \dots, n$, and with volume at least δ. In fact, these boxes are the minimal feasible solutions of the inequality $\sum_{i=1}^n \log |x_i| \geq \log \delta$, over the lattice $\mathcal{L}$. If $\mathcal{F}$ is the family of all minimal feasible solutions to this inequality, then, as was done in Theorem 10 and Corollary 3, one can use Lemma 3 to prove that

$$|\mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{F})| \leq \sum_{i=1}^{n} |\mathbb{P}_i||\mathcal{X}|$$

for any non-empty subset $\mathcal{X} \subseteq \mathcal{F}$. Thus all minimal such boxes with volume at least δ can be generated in quasi-polynomial time.

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